Copies of the inside front and back covers of the Griffiths text are provided on the last page.

1. In class, we used:

$$W = \frac{1}{2} \int \sigma V \, da$$

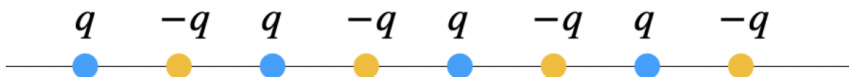to show that the energy of a uniformly-charged spherical shell is equal to:

$$W = \frac{1}{8\pi\varepsilon_0} \frac{q^2}{R}.$$

Repeat this calculation starting from:

$$W = \frac{\varepsilon_0}{2} \int_{\substack{\text{all} \\ \text{space}}} E^2 \, d\tau.$$

Of course, you should get the same result.

2. Consider an infinite chain of point charges, $\pm q$ (with alternating signs), strung out along the x-axis, each a distance a from its nearest neighbour.
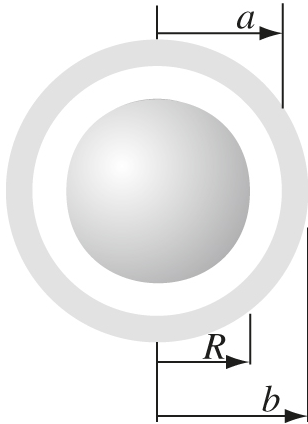

Show that the work per particle required to assemble this system is given by:

$$W = -\frac{\alpha}{4\pi\varepsilon_0} \frac{q^2}{a}$$

where α is a dimensionless constant known as the Madelung constant. Determine the numerical value of α.

3. A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.

(a) Find the surface charge density σ at $r = R$, $r = a$, and $r = b$.

(b) Find the potential at the centre, using infinity a the reference point.

(c) Now the outer surface is touched to a grounding wire, which drains off the charge and lowers its potential to zero. How do the answers to (a) and (b) change?

4. Two large metal plates (each of area A) are held a small distance d apart. Suppose we put a charge Q on each plate; what is the electrostatic pressure on the plates?

5. Assume that the electric field in some region is given (in spherical coordinates) by the expression:

$$\mathbf{E}(\mathbf{r}) = \frac{k}{r} \left[3\,\hat{r} + 2\sin\theta\cos\theta\sin\phi\,\hat{\theta} + \sin\theta\cos\phi\,\hat{\phi} \right],$$

where k is a constant. What is the corresponding charge density $\rho(\mathbf{r})$?

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \dfrac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \dfrac{\partial t}{\partial z}\,\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left(\dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial y^2} + \dfrac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi}$

Curl: $\quad \nabla \times \mathbf{v} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\phi) - \dfrac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$

$$+ \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(r v_\theta) - \dfrac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial t}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\,\dfrac{\partial t}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \dfrac{1}{s}\dfrac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \dfrac{\partial t}{\partial z}\,\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{s}\dfrac{\partial}{\partial s}(s v_s) + \dfrac{1}{s}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left[\dfrac{1}{s}\dfrac{\partial v_z}{\partial \phi} - \dfrac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\dfrac{\partial v_s}{\partial z} - \dfrac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \dfrac{1}{s}\left[\dfrac{\partial}{\partial s}(s v_\phi) - \dfrac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s\dfrac{\partial t}{\partial s}\right) + \dfrac{1}{s^2}\dfrac{\partial^2 t}{\partial \phi^2} + \dfrac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\quad \nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\quad \nabla \times (\nabla f) = 0$

(11) $\quad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\quad \int_{\mathbf{a}}^{\mathbf{b}}(\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\quad \int(\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\quad \int(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\[2mm] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \nabla \cdot \mathbf{B} = 0 \\[2mm] \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\[2mm] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \nabla \cdot \mathbf{B} = 0 \\[2mm] \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\[2mm] \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\[2mm] \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy:
$$U = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau$$

Momentum:
$$\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B}) \, d\tau$$

Poynting vector:
$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

Larmor formula:
$$P = \frac{\mu_0}{6\pi c} q^2 a^2$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12} \, \text{C}^2/\text{Nm}^2$ (permittivity of free space)

$\mu_0 = 4\pi \times 10^{-7} \, \text{N/A}^2$ (permeability of free space)

$c = 3.00 \times 10^8 \, \text{m}/s$ (speed of light)

$e = 1.60 \times 10^{-19} \, \text{C}$ (charge of the electron)

$m = 9.11 \times 10^{-31} \, \text{kg}$ (mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \sin\theta \cos\phi \, \hat{\mathbf{r}} + \cos\theta \cos\phi \, \hat{\boldsymbol{\theta}} - \sin\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta \sin\phi \, \hat{\mathbf{r}} + \cos\theta \sin\phi \, \hat{\boldsymbol{\theta}} + \cos\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta \, \hat{\mathbf{r}} - \sin\theta \, \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z \right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta \cos\phi \, \hat{\mathbf{x}} + \sin\theta \sin\phi \, \hat{\mathbf{y}} + \cos\theta \, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta \cos\phi \, \hat{\mathbf{x}} + \cos\theta \sin\phi \, \hat{\mathbf{y}} - \sin\theta \, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi \, \hat{\mathbf{x}} + \cos\phi \, \hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s \cos\phi \\ y = s \sin\phi \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \cos\phi \, \hat{\mathbf{s}} - \sin\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi \, \hat{\mathbf{s}} + \cos\phi \, \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{s}} = \cos\phi \, \hat{\mathbf{x}} + \sin\phi \, \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi \, \hat{\mathbf{x}} + \cos\phi \, \hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$